Walk-Sums and Gaussian BP (#210)
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Walk-sum framework for Gaussian inference, new sufficient conditions for BP convergence.

**Information form** of the Gaussian density:

\[ p(x_1, ..., x_n) \propto \exp\left\{-\frac{1}{2}x^T J x + h^T x\right\} \]

\( J \) is sparse:

\[ G = (V, E) \]

Define edge weights \( \rho_{ij} = -\frac{J_{ij}}{\sqrt{J_{ii}J_{jj}}} \).

Given a walk \( w \) in \( G \), let \( \rho(w) = \prod_{(i,j) \in w} \rho_{ij} \).

**Walk-summable** if spectral radius \( \rho(|R|) < 1 \), \( R \) is matrix of edge weights.*

\[ \text{cov}(x_i, x_j) = \sum_{w:i\rightarrow j} \rho(w), \quad \text{mean}(x_i) = \sum_{w:*\rightarrow i} h \ast \rho(w) \]

*Includes trees, attractive models and diagonally dominant models.
Walk-Sum View of Belief Propagation

BP in trees ≡ recursive walk-sum calculation:

Loopy BP on $G$ is equivalent to computing exact walk-sums in the computation tree:

WS on $G$ ⇒ BP converges: means correct, variances → sums over backtracking walks.

A tighter condition is WS on the comp. tree:

(i) $\varrho_\infty = \lim_{n \to \infty} \varrho(|R_n|) \leq \varrho(|R|)$.
(ii) $\varrho_\infty < 1$ ⇒ BP variances converge.
(iii) $\varrho_\infty > 1$ ⇒ invalid computation tree.