A LYAPUNOV METHOD FOR ESTABLISHING STRONG AND
WEAK STABILITY OF INFINITE-DIMENSIONAL KALMAN
FILTERS FOR SPACE-TIME ESTIMATION

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Abstract. This paper examines the stability of discrete-time, infinite-dimensional Kalman filters
used for space-time estimation. In particular, conditions are established for the unforced trajec-
tories of the filter to converge to zero in either the strong or weak topologies. This is accomplished
by developing a general Lyapunov theory for time-varying infinite-dimensional systems that provides
sufficient conditions for such convergence. In the strong case, the conditions are also shown to be
necessary when restricted to trajectories that are square summable.

Key words. stability theory, linear time-varying infinite-dimensional systems, strong stability,
weak stability, Lyapunov function, Kalman filter

AMS subject classifications. 93D30, 93E11

1. Introduction. This paper studies the stability of a class of discrete-time,
infinite-dimensional, time-varying Kalman filters. Existing results concerning finite-
dimensional Kalman filter stability [7, 8, 15] and generalizations to infinite dimen-
sions [9] assume that reachability and observability Gramians are uniformly positive
definite1 so as to establish exponential stability. The results in this paper establish
weaker forms of stability when the reachability and observability Gramians may not
be uniformly positive definite.

In particular, we are able to prove strong stability of the filter and strong square
summability of the trajectories. By strong stability and strong square summability,
we mean that \( z(t) \) converges strongly to 0,

\[
\lim_{t \to \infty} \|z(t)\| = 0, \quad (1.1)
\]

and

\[
\sum_{t=0}^{\infty} \|z(t)\|^2 < \infty, \quad (1.2)
\]

respectively, where \( z(t) \) follows the Kalman filter dynamics. The main result is stated
as Theorem 5.1. Although this theorem imposes less stringent conditions on the
reachability Gramian, the conditions on the observability Gramians are similar to
those required in previous investigations of exponential stability [7–9, 15]. When
relaxing the conditions on the observability Gramian, we are able to prove weak
stability of the filter. By weak stability, we mean that \( z(t) \) converges weakly to 0,

\[
\lim_{t \to \infty} \langle z(t), z' \rangle = 0 \quad \forall z' \in X, \quad (1.3)
\]

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\footnote{1A symmetric operator \( A \) is uniformly positive definite if there exists a constant \( \alpha \) such that \( \langle v, Av \rangle > \alpha \|v\|^2 \) for all vectors \( v \neq 0 \).}
where, again, \( z(t) \) follows the Kalman filter dynamics. The main result on weak stability is Theorem 7.1.

Others have examined forms of stability for general infinite-dimensional linear systems that are weaker than exponential stability. In [3, 4], Curtain and Oostveen are concerned with linear time-invariant state-space models and relating a notion of stability to properties of the model such as stabilizability and detectability. Although they are also concerned with a notion of stability weaker than exponential, their notion is more restrictive than the weak or strong stability conditions considered here. Przulj and Rolewicz consider many forms of time-varying, infinite-dimensional system stability and the interrelationships among them [25–28]. However, they do not consider the notions of strong or weak stability as they are used in this paper.

The proofs of our results are extensions of those given in [7, 8, 15] for the finite-dimensional case rather than those in [9] for the infinite-dimensional case. In particular, we make use of Lyapunov functions. This has necessitated the development of an appropriate Lyapunov theory for establishing strong and weak stability of time-varying infinite-dimensional linear systems.

Many others have extended various aspects of finite-dimensional Lyapunov theory [17, 18] to infinite-dimensional dynamical systems. Some references include [1, 5, 6, 10, 19–24, 29]. However only two of these references consider strong or weak stability. In particular, Ross states a theorem concerning weak stability of time-invariant linear systems [29, Theorem 2.1.2]. This theorem does not apply to the dynamics of the Kalman filter because of the theorem’s assumption concerning the time-invariance of the system whose stability is in question. However, our work on time-varying systems is motivated, in part, by his work. Massera and Schäffer state a theorem concerning strong stability of time-varying systems [22, Theorem 4.1]. This theorem does not apply to the case under consideration for a more subtle reason than for Ross’s theorem. Specifically, Massera and Schäffer require that the Lyapunov function, \( V(x, t) \), have what they term “an infinitely small upper bound”. Their definition states that a continuous function \( a \) is an infinitely small upper bound for a Lyapunov function \( V(x, t) \) if

\[
V(x, t) \leq a(||x||) \quad \forall x, t. \tag{1.4}
\]

The existence of the infinitely small upper bound guarantees that the Lyapunov function is bounded over every closed ball centered at the origin and that the bound changes continuously with the radius of the ball. We will need to relax this restriction.

Our analysis of the stability issues has been motivated by a study of Kalman filters used to process satellite remote sensing data, especially in the meteorology and oceanography communities. A typical problem in this area is the determination of sea surface height from satellite altimeter measurements taken pointwise along an orbit that only sparsely covers the ocean [32]. One can view this as a discrete-time, space-time estimation problem in which one is interested in estimating sea surface height from a set of data taken over the period of the satellite. The desire for computationally efficient implementations of the Kalman filter for space-time problems has led to the development of a variety of approximation techniques [2, 11–14, 16, 32–34]. Understanding the stability of the exact Kalman filter is necessary to understand the behavior of the approximations. The standard stability results do not apply in many geophysical situations because the reachability Gramian is not uniformly positive definite (a consequence of the driving noise being spatially smooth).
We develop our stability ideas more fully over the next few sections. The next section sets up the mathematical framework. In Section 3, theorems regarding the boundedness of the Kalman filter error covariances are stated and proved. Then, a Lyapunov theory for strong stability is developed in Section 4. This is applied to demonstrate strong stability of the filter in Section 5. Next, a Lyapunov theory for weak stability is developed in Section 6 and applied in Section 7 to demonstrate weak stability of the filter under more relaxed restrictions than those in Section 5.

2. The Framework. In this paper, the states $x(t)$ and measurements $y(t)$ of a discrete-time system take values in Hilbert spaces $X$ and $Y$, respectively.\(^2\) The dynamics of the state are given by

$$x(t + 1) = A(t)x(t) + w(t), \quad (2.1)$$

and the measurements are of the form

$$y(t) = C(t)x(t) + n(t) \quad (2.2)$$

where $w(t)$ is a stochastic forcing with covariance $\Lambda_w(t)$, $n(t)$ is additive measurement noise with covariance $\Lambda_n(t)$, and $x(0)$ is an initial condition with mean $\mu$ and covariance $\Lambda_x$. The operators $A(t) : X \mapsto X$, $C(t) : X \mapsto Y$, $\Lambda_n(t) : Y \mapsto Y$, $\Lambda_w(t) : X \mapsto X$, and $\Lambda_x : X \mapsto X$ are all assumed to be bounded linear mappings. In addition, the operator $\Lambda_n(t)$ is assumed to have a bounded inverse for each $t$. Also, the notation $\Phi(t, s)$ is used to denote the transition map associated with (2.1). That is

$$\Phi(t, s) = A(t - 1)A(t - 2) \cdots A(s). \quad (2.3)$$

The filtering equations follow $[9, \text{p. 297}].$ Recall that the Kalman filter recursively computes a sequence of estimates of $x(t)$ given data up to time $t$, $\hat{x}(t|t)$, termed *updated estimates*; another sequence of estimates of $x(t)$ but given data up to time $t - 1$, $\hat{x}(t|t - 1)$, termed *predicted estimates*; and the associated error covariances $\Lambda_e(t|t)$ and $\Lambda_e(t|t - 1)$. The recursion is a two-step procedure, involving an update and prediction step at each point in time. The update is typically written in terms of the covariance

$$\Lambda_n(t) = C(t)\Lambda_n(t|t - 1)C^*(t) + \Lambda_n(t). \quad (2.4)$$

of $\nu(t) = y(t) - C(t)\hat{x}(t|t - 1)$, which is the residual in the predicted measurement given data up to time $t - 1$. Specifically, the update takes the form

$$\hat{x}(t|t) = \hat{x}(t|t - 1) + \Lambda_e(t|t - 1)C^*(t)\Lambda_e^{-1}(t)(y(t) - C(t)\hat{x}(t|t - 1)) \quad (2.5)$$

$$\Lambda_e(t|t) = \Lambda_e(t|t - 1) - \Lambda_e(t|t - 1)C^*(t)\Lambda_e^{-1}(t)C(t)\Lambda_e(t|t - 1), \quad (2.6)$$

and the predict step is given by

$$\hat{x}(t + 1|t) = A(t)\hat{x}(t|t) \quad (2.7)$$

$$\Lambda_e(t + 1|t) = A(t)\Lambda_e(t|t)A^*(t) + \Lambda_w(t). \quad (2.8)$$

\(^2\)The Hilbert spaces in this paper are analyzed using two different topologies, strong and weak, and associated notions of convergence $[30, \text{Section 3.11}].$ A sequence $\{u_n\}$ is said to converge in the strong topology, or strongly, to a point $u$, if $\lim_{n \to \infty} \|u_n - u\| = 0$ where $\| \cdot \|$ is the standard norm induced by the inner product. A sequence $\{u_n\}$ is said to converge in the weak topology, or weakly, to a point $u*$ if $\lim_{n \to \infty} (u, u_n - u) = 0$ for all vectors $u$ in Hilbert space where $(\cdot, \cdot)$ is the inner product. The discussion in this paper also makes use of a notion of positive definiteness. Specifically, a symmetric operator $M$ is positive definite if $(u, Mu) > 0$ for all vectors $u \neq 0$ in the Hilbert space.
These recursions are initialized with \( \hat{x}(0|0) = \mu \) and \( \Lambda_e(0|0) = \Lambda_x \), the prior mean and covariance of \( x \), respectively. Of principal interest is the stability of the recursion for the updated estimates

\[
\hat{x}(t|t) = (I - \Lambda_e(t|t-1)C^*(t)\Lambda^{-1}_x(t)C(t))A(t)\hat{x}(t-1|t-1) + \Lambda_e(t|t-1)C^*(t)\Lambda^{-1}_x(t)y(t),
\]

and, in particular, the unforced dynamics, which are given by

\[
z(t) = [I - \Lambda_e(t|t-1)C^*(t)\Lambda^{-1}_x(t)C(t)]A(t)z(t-1).
\]

The stability results discussed in this paper make various assumptions concerning certain reachability and observability Gramians. The specific reachability and observability Gramians of interest are given by

\[
\mathcal{R}(t,s) \triangleq \sum_{\tau=s}^{t-1} \Phi(t,\tau+1)\Lambda_m(\tau)\Phi^*(t,\tau+1),
\]

and

\[
\mathcal{I}(t,s) \triangleq \sum_{\tau=s}^{t} \Phi(\tau,t)C^*(\tau)\Lambda^{-1}_m(\tau)C(\tau)\Phi(\tau,t)
\]

respectively.

3. Boundedness of the Error Covariances. The first step in proving the stability of the Kalman filter is to bound the update error covariance. The following two theorems provide such bounds given conditions on the reachability and observability Gramians, (2.11) and (2.12). The development follows that of [7,8].

**Theorem 3.1.** Suppose there exist constants \( \alpha_1, \alpha_2, \beta_1, \beta_2, T \geq 0 \) such that

\[
\alpha_1 I \leq \mathcal{R}(t,t-T) \leq \alpha_2 I \quad \forall t \geq T \tag{3.1}
\]

\[
\beta_1 I \leq \mathcal{I}(t,t-T) \leq \beta_2 I \quad \forall t \geq T. \tag{3.2}
\]

Then, the update error covariance of the Kalman filter satisfies

\[
\Lambda_e(t|t) \leq \left( \frac{1}{\beta_1} + T \frac{\beta_2 \alpha_2}{\beta_1^2} \right) I. \tag{3.3}
\]

A proof of the upper bound in Theorem 3.1 for finite-dimensional systems is given in [7]. The proof extends to the infinite-dimensional setting here, without modification. The next theorem provides a lower bound on the update error covariance. The theorem statement is a modification of that in [8] that takes into account the errors cited in [7]. Moreover, the following theorem makes explicit the assumption that \( A(t) \) and its inverse are bounded.

**Theorem 3.2.** Suppose that \( \forall t, \Lambda_m(t) \) and \( \Lambda_x \) have bounded inverses. Moreover, suppose that there exist constants \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, T \geq 0 \) such that

\[
\alpha_1 I \leq \mathcal{R}(t,t-T-1) \leq \alpha_2 I \quad \forall t \geq T + 1 \tag{3.4}
\]

\[
\beta_1 I \leq \mathcal{I}(t-1,t-T) \leq \beta_2 I \quad \forall t \geq T \tag{3.5}
\]

\[
||A(t)^{-1}|| \leq \frac{1}{\gamma_1} \quad \forall t \tag{3.6}
\]

\[
||A(t)|| \leq \gamma_2 \quad \forall t. \tag{3.7}
\]
Then, the update error covariance of the Kalman filter satisfies

\[ \Lambda_e(t|t) \geq \left( \frac{\alpha^2_1 \gamma_1}{\alpha_1 \gamma_1 + T \alpha_2 \beta_2} + \beta_2 \right) I. \quad (3.8) \]

A detailed proof of Theorem 3.2 is given in the appendix.

4. Lyapunov Theory for Strong Stability. In finite dimensions, the Lyapunov function used to establish exponential stability of the Kalman filter dynamics is the quadratic form associated with the inverse of the update error covariance, \( \langle z(t), \Lambda^{-1}(t|t)z(t) \rangle \). One would like to use this same, natural Lyapunov function to establish stability for space-time filtering problems. For these problems, however, \( \Lambda^{-1}(t|t) \) may be an unbounded operator. Intuitively, \( \langle z(t), \Lambda^{-1}(t|t)z(t) \rangle \) should still work as a Lyapunov function, however, provided one can establish descent along trajectories.

The general situation is illustrate in Figure 4.1. The figure shows coordinate slices of increasing curvature for a quadratic form associated with an unbounded operator. Consider evaluating the form along trajectories of a Kalman filter. Since the curvature of coordinate slices may become arbitrarily large, the quadratic form may take on arbitrarily large values as the initial states of the filter vary, even if they are constrained to lie in a bounded set. Thus, given only that the successive differences of the quadratic form along trajectories are bounded from below, one can not expect that the state will converge to zero at a uniform rate across all initial conditions in a bounded set. However, the state may still be guaranteed to tend to zero if the quadratic form is decreasing fast enough along trajectories. In particular, we consider two different possibilities for how the trajectory \( z(t) \) may tend to zero, in this section.

**Definition 4.1.** A trajectory \( z(t) \) is strongly stable if

\[ \lim_{t \to \infty} \| z(t) \| = 0. \quad (4.1) \]

**Definition 4.2.** A trajectory \( z(t) \) is strongly square summable if

\[ \sum_{t=0}^{\infty} \| z(t) \|^2 < \infty. \quad (4.2) \]

One can also consider the behavior of all trajectories for a system, as follows.
Definition 4.3. A dynamical system is strongly stable if the unforced trajectories \( z(t) \) satisfy
\[
\lim_{t \to \infty} \|z(t)\| = 0
\]
for all initial conditions.

Definition 4.4. A dynamical system is strongly square summable if the unforced trajectories \( z(t) \) satisfy
\[
\sum_{t=0}^{\infty} \|z(t)\|^2
\]
for all initial conditions.

The following theorem characterizes individual trajectories. The Kalman filter, as a system, is analyzed in the next section. One method for verifying whether an individual trajectory \( z(t) \) is strongly stable or strongly square summable is to find a sequence of trajectories \( z_\sigma(t) \) which converge to \( z(t) \) and for which one can establish exponential stability using quadratic Lyapunov functions. This idea is made more precise in the following theorem.

Theorem 4.5. Let \( X \) be a real Hilbert space, and \( \mathcal{B}(X) \), the space of bounded linear operators on \( X \).

Let \( z(t) \in X \) evolve according to
\[
z(t+1) = F(t)z(t)
\]
with \( F(t) \in \mathcal{B}(X) \). Consider a family of approximations to \( z(t) \), \( z_\sigma(t) \in X \) for \( \sigma \in \mathbb{R}^+ \), evolving according to
\[
z_\sigma(t+1) = F_\sigma(t)z(t)
\]
with \( z_\sigma(0) = z(0) \) and \( F_\sigma(t) \in \mathcal{B}(X) \) and converging pointwise in time for all \( t \),
\[
\lim_{\sigma \to \infty} \|z_\sigma(t) - z(t)\| = 0 \quad \forall t.
\]

Now, let \( U_\sigma(t) \in \mathcal{B}(X) \) be a family of symmetric, positive definite operators, and let
\[
V_\sigma(z_\sigma(t),t) = \langle z_\sigma(t), U_\sigma(t)z_\sigma(t) \rangle
\]
be the associated quadratic forms.

Suppose there exists constants \( T \) and \( \eta \), symmetric operators \( W_\sigma(t) \in \mathcal{B}(X) \), and a subset \( S \subset X \) such that
1. \( V_\sigma(z_\sigma(t+1),t+1) - V_\sigma(z_\sigma(t),t) \leq 0 \) and
2. \( V_\sigma(z_\sigma(t),t) - V_\sigma(z_\sigma(t-T),t-T) \leq -\langle z_\sigma(t), W_\sigma(t)z_\sigma(t) \rangle \quad \forall t \geq T \)
3. \( \liminf_{\sigma \to \infty} V_\sigma(z_\sigma(0),0) \leq \infty \quad \forall z_\sigma(0) \in S \).

Then, the trajectories \( z(t) \) are strongly square summable and strongly stable for initial conditions \( z(0) \in S \).

Remarks. Note that the first two conditions on \( V_\sigma, W_\sigma \), and \( z(t) \) establish that \( V_\sigma \) are Lyapunov functions for \( z_\sigma(t) \) which establish exponential stability of the dynamics.
The third condition ensures that the $V_{\sigma}$, evaluated at time zero, are reasonably well behaved as $\sigma$ increases.

Proof. Note that for any $s \in [0, T)$,

$$0 \leq V_{\sigma}(z_{\sigma}(t), t) \leq V_{\sigma}(z_{\sigma}(0), 0) - \sum_{\tau = 1}^{\left\lfloor \frac{t}{\tau} \right\rfloor} \left\langle z_{\sigma}(\tau T + s), W_{\sigma}(\tau T + s)z_{\sigma}(\tau T + s) \right\rangle \quad \forall t.$$  \hfill (4.9)

Thus,

$$\eta \sum_{\tau = 1}^{\infty} \|z_{\sigma}(\tau T + s)\|^2 \leq V_{\sigma}(z_{\sigma}(0), 0).$$  \hfill (4.10)

By Fatou's lemma [35, Theorem 10.29],

$$\eta \sum_{\tau = 1}^{\infty} \liminf_{\sigma \to \infty} \|z_{\sigma}(\tau T + s)\|^2 \leq \eta \liminf_{\sigma \to \infty} \sum_{\tau = 1}^{\infty} \|z_{\sigma}(\tau T + s)\|^2 \leq \eta \liminf_{\sigma \to \infty} V_{\sigma}(z_{\sigma}(0), 0).$$  \hfill (4.11)

Thus,

$$\eta \sum_{\tau = 1}^{\infty} \|z(\tau T + s)\|^2 < \infty$$  \hfill (4.12)

and

$$\sum_{t = 0}^{\infty} \|z(t)\|^2 < \infty \quad \forall z(0) \in S.$$  \hfill (4.13)

\[\square\]

Theorem 4.5 also has a partial converse. Specifically, if the trajectories $z(t)$ are square summable for a set of initial conditions, there exists not only a sequence $z_{\sigma}(t)$ approximating $z(t)$ but also functions $V_{\sigma}$, operators $W_{\sigma}$, and constant $\eta$ satisfying the three conditions of Theorem 4.5. The following theorem states this precisely.

**Theorem 4.6.** Let $z(t) \in X$ evolve according to

$$z(t + 1) = F(t)z(t)$$  \hfill (4.14)

with $F(t) \in B(X)$. Suppose that there exists a set $S \subset X$ such that

$$\sum_{\tau = 0}^{\infty} \|z(\tau)\|^2 < \infty \quad \forall z(0) \in S.$$  \hfill (4.15)

Then, there exists a sequence of trajectories $z_{\sigma}(t)$ and a sequence of functions $V_{\sigma} : X \times \mathbb{R} \to \mathbb{R}$ such that:

1. $z_{\sigma}(0) = z(0) \quad \forall \sigma$
2. $\lim_{\sigma \to \infty} \|z_{\sigma}(t) - z(t)\| = 0 \quad \forall t$
3. $V_{\sigma} \geq 0$
4. $V_{\sigma}(z_{\sigma}(t + 1), t + 1) - V_{\sigma}(z_{\sigma}(t), t) \leq -\|z_{\sigma}(t)\|^2$
5. $\lim_{\sigma \to \infty} V_{\sigma}(z_{\sigma}(0), 0) < \infty \quad \forall z(0) \in S.$
Remarks. The first two conditions establish that $z_\sigma(t)$ approximates $z(t)$ point-wise and has the same initial condition. The third and fourth conditions imply that for each $\sigma$, $V_\sigma$ is a Lyapunov function for $z_\sigma$ that establishes exponential stability of the evolution of $z_\sigma$. The last condition ensures that $V_\sigma$ is reasonably well behaved across $\sigma$.

Proof. The proof is constructive. Specifically, define a sequence of trajectories

$$z_\sigma(t) = \begin{cases} z(t) & t \leq \sigma \\ 0 & t > \sigma, \end{cases}$$  \hspace{1cm} (4.16)$$
a sequence of functions

$$V_\sigma(z_\sigma(t),t) = \sum_{\tau=t}^{\infty} \||z_\sigma(\tau)||^2$$  \hspace{1cm} (4.17)$$
and a limiting function

$$V(z(0),0) = \sum_{\tau=0}^{\infty} ||z(\tau)||^2.$$  \hspace{1cm} (4.18)$$

Then,
1. $z_\sigma(0) = z(0) \forall \sigma$
2. $\lim_{\tau \to \infty} ||z_\sigma(t) - z(t)|| = 0 \forall t$
3. $V_\sigma \geq 0$
4. $V_\sigma(z_\sigma(t+1),t+1) - V_\sigma(z_\sigma(t),t) = -||z_\sigma(t)||^2$
5. $\lim_{\tau \to \infty} V_\sigma(z_\sigma(0),0) = V(z(0)) < \infty \forall z(0) \in S.$

Although Theorem 4.6 is not subsequently applied to the study of Kalman filter stability, it is important by itself since it establishes that the conditions in Theorem 4.5 are necessary when restricted to the case of strong summability. Theorem 4.5 is readily applicable to the study of Kalman filter stability, however.

Theorem 4.5 can be used to prove stability of the Kalman filter. The precise notions of stability are those given in Definitions 4.3 and 4.4.

Theorem 5.1. Suppose that the dynamics matrix is bounded above and below, i.e. there exist constants $\gamma_1, \gamma_2 > 0$ such that

$$||A(t)^{-1}|| \leq \frac{1}{\gamma_1} \forall t$$  \hspace{1cm} (5.1)$$

$$||A(t)|| \leq \gamma_2 \forall t.$$  \hspace{1cm} (5.2)$$

and suppose that the system is uniformly observable, i.e. there exist constants $\beta_1, \beta_2, T > 0$ such that

$$\beta_1 I \leq I(t,t-T) \leq \beta_2 I \forall t \geq T$$  \hspace{1cm} (5.3)$$

Moreover, suppose that the measurement noise covariance is bounded below, i.e. there exists a constant $\sigma^2 > 0$ such that

$$\Lambda_n(t) \geq \sigma^2 I \forall t.$$  \hspace{1cm} (5.4)$$
Then, the dynamics of the Kalman filter (2.10) are strongly square summable and strongly stable, i.e.

$$\sum_{t=0}^{\infty} \| z(t) \|^2 < \infty \quad (5.5)$$

and

$$\lim_{t \to \infty} \|z(t)\| = 0 \quad (5.6)$$

for all initial conditions where $z(t)$ obeys the filter dynamics.

**Proof.** The proof of Theorem 5.1 primarily involves constructing a sequence of systems so that Theorem 4.5 can be applied.

Specifically, consider adding white noise to the process noise and initial covariance, thereby shifting the covariances by a multiple of the identity. For a shift of $(1/\sigma^2)I$ with $\sigma^2 \geq 1$, the new covariances will be

$$\Lambda'_w(t) = \Lambda_w(t) + \frac{1}{\sigma^2} I \quad (5.7)$$

and

$$\Lambda'_x = \Lambda_x + \frac{1}{\sigma^2} I. \quad (5.8)$$

For this new system, all other matrices remain unaltered, i.e. $C'(t) = C(t), \Lambda'_n(t) = \Lambda_n(t), A'(t) = A(t)$. Let $z_{\sigma}(t)$ be a sequence of states propagating according to the filter dynamics for the new system. Then, for each time point $t$,

$$\lim_{\sigma \to \infty} \| z_{\sigma}(t) - z(t) \| = 0 \quad (5.9)$$

if $z_{\sigma}(0) = z(0)$ by the continuous dependence on $\sigma$ of the filtering dynamics (2.10).

Now, since $C'(t) = C(t), \Lambda'_n(t) = \Lambda_n(t), A'(t) = A(t)$, the observability gramian of the new system, $T'_\sigma(t,s)$, is the same as the original, $T(t,s)$. By the assumptions of Theorem 5.1,

$$\beta_1 I \leq T'_\sigma(t,t-T) \leq \beta_2 I \quad \forall t \geq T. \quad (5.10)$$

The reachability gramian of the new system is also bounded above and below due to the shift (5.7). Specifically, there exists a constant $\alpha_2$ such that $\forall t \geq T$.

$$\frac{1}{\sigma^2} I \leq R'_\sigma(t,t-T) \quad (5.11)$$

$$= \sum_{\tau=t-T}^{t-1} \Phi'(t,\tau+1) \Lambda'_w(\tau) (\Phi')^*(t,\tau+1) \quad (5.12)$$

$$\leq \sum_{\tau=t-T}^{t-1} \Phi(t,\tau+1) \Lambda'_w(\tau) (\Phi')^*(t,\tau+1) \quad (5.13)$$

$$\leq \alpha_2 I. \quad (5.14)$$

Moreover, since

$$R'_\sigma(t+1, t-T) = A(t) \sum_{\tau=t-T}^{t-1} \Phi(t,\tau+1) \Lambda'_w(\tau) (\Phi')^*(t,\tau+1) A^*(t) \quad (5.15)$$
\[ R'_\sigma(t + 1, t - T) \] can be bounded as follows:

\[
\frac{\gamma_1^2}{\sigma^2} I \leq R'_\sigma(t + 1, t - T) \leq \gamma_2^2 \sigma I \quad \forall t \geq T. \tag{5.16}
\]

By Theorems 3.1 and 3.2, the update error covariance of the modified system, \( \Lambda'_{e,\sigma}(t|t) \), is bounded above and below as follows:

\[
\left( \frac{\gamma_1^5}{\sigma^2 \gamma_1^3 + (T + 1) \gamma_2^2 \beta_2 \beta_2} + \beta_2 \right) I \leq \Lambda'_{e,\sigma}(t|t) \leq \left( \frac{1}{\beta_1} + T \frac{\gamma_2^2 \beta_2 \gamma_2}{\beta_1^2} \right) I \quad \forall t \geq T.
\tag{5.17}
\]

Hence, one can consider using

\[ V'_\sigma(x, t) = \langle x, (\Lambda'_{e,\sigma}(t|t))^{-1} x \rangle \quad t \geq T \tag{5.18} \]

to establish stability of the Kalman filter.

In order to use this sequence of systems to establish strong stability of the original filter dynamics (2.10), one needs to verify each of the three enumerated conditions of Theorem 4.5 for \( U'_\sigma(t) = (\Lambda'_{e,\sigma}(t|t))^{-1}, \hat{U} = \Lambda^{-1}_\sigma(T|T) \), and \( W'_\sigma(t) = I(t, t - T) \). The verification follows. Note that in what follows, the trajectories are examined starting at time \( T \), not 0, since \( V'_\sigma \) is only to find after time \( T \).

1. In [8, p. 764], a bound on the differences of \( V'_\sigma(z, t|t) \) is established in finite dimensions. The derivation of the bound holds in general Hilbert spaces. The bound states that

\[
V'_\sigma(z, t + 1), t + 1) - V'_\sigma(z, t), t) \leq 0 \tag{5.19}
\]

\[
V'_\sigma(z, t), t) - V'_\sigma(z, t - T), t - T) \leq \sum_{\tau = T}^{t} \langle z(\tau), C^*(\tau)\Lambda^{-1}_n(\tau)C(\tau)z(\tau) \rangle. \tag{5.20}
\]

Since

\[
\sum_{\tau = T}^{t} \langle z(\tau), C^*(\tau)\Lambda^{-1}_n(\tau)C(\tau)z(\tau) \rangle = \sum_{\tau = T}^{t} \langle \Phi(\tau, t)z(t), C^*(\tau)\Lambda^{-1}_n(\tau)C(\tau)\Phi(\tau, t)z(t) \rangle = \langle z(t), I(t, t - T)z(t) \rangle, \tag{5.21}
\]

one has that

\[
V'_\sigma(z, t), t) - V'_\sigma(z, t - T), t - T) \leq \langle z(t), I(t, t - T)z(t) \rangle. \tag{5.22}
\]

2. \( \langle z, I(t, t - T)z(t) \rangle \geq \beta_1||z, t||^2 \) for all \( \sigma \) and \( t \geq T \) by (5.3).

3. Finally, \( \lim_{T \to \infty} z(T), (\Lambda'_{e,\sigma}(T|T))^{-1} z(T) \) for all initial conditions \( z(0) \). The limit follows simply from the fact that the estimator is a continuous function of \( \sigma \). That \( z(T) \in D(\Lambda^{-1}_\sigma(T|T)) \) follows from the fact that \( z(T) \in R(\Lambda_{e,\sigma}(T|T - 1)) \) by (2.10) and that \( \Lambda^{-1}_\sigma(T|T) = \Lambda^{-1}_\sigma(T|T - 1) + C^*(T)\Lambda^{-1}_n(T)C(T) \) where

\[
C^*(T)\Lambda^{-1}_n(T)C(T) \leq I(T, 0) \leq \beta_2 I. \tag{5.23}
\]

Now, the desired result follows from Theorem 4.5. \qed
6. **Lyapunov Theorem for Weak Stability.** In this section we investigate guarantees on weak stability of trajectories, defined as follows.

**Definition 6.1.** A trajectory \( z(t) \) is weakly stable if

\[
\lim_{t \to \infty} \langle z', z(t) \rangle = 0 \quad \forall z' \in X.
\]  

(6.1)

There exists a corresponding definition of weakly stable systems.

**Definition 6.2.** A dynamical system is weakly stable if the unforced trajectories \( z(t) \) satisfy

\[
\lim_{t \to \infty} \langle z', z(t) \rangle = 0 \quad \forall z' \in X
\]

for all initial conditions.

In the next section, we consider the weak stability of the Kalman filter, as a system. Here, we develop a Lyapunov theory for weak stability of trajectories that is a relaxed version of Theorem 4.5. The specific condition that is relaxed is Condition 2, which restricts \( W_\sigma \) to be uniformly positive definite. This condition is relaxed to \( W_\sigma \) being only positive definite. This is motivated by Ross’s work on using Lyapunov functions to establish weak stability of time-invariant systems [29, Theorem 2.1.2].

Figure 6.1 provides some intuition as to why \( W_\sigma \) being positive definite is sufficient for establishing weak stability but not strong stability of a given system. The figure depicts the quadratic form associated with \( W_\sigma \), which provides a lower bound on the negative magnitude of successive differences of a quadratic form \( V_\sigma \) evaluated along trajectories. Each graph in Figure 6.1 plots a different coordinate slice of the quadratic function. That the function is positive away from the origin implies that each coordinate of the state of the given system, \( z(t) \), is converging to zero. Thus, one expects \( z(t) \) to converge weakly to 0. However, the curvature of each slice may be arbitrarily small so that \( z(t) \) may not be converging strongly to 0.

These ideas are made precise in the following theorem. The statement is almost the same as that of Theorem 4.5. The primary difference is that the lower bound on \( W_\sigma(t) \) is replaced by a weaker condition in terms of the ranges of the limiting operator \( W(t) \).

**Theorem 6.3.** Let \( X \) be a real Hilbert space, and \( B(X) \), the set of bounded linear operators on \( X \).

Let \( z(t) \in X \) evolve according to

\[
z(t + 1) = F(t)z(t)
\]

(6.3)
with $F(t) \in \mathcal{B}(X)$. Consider a family of approximations to $z(t)$, $z_\sigma(t) \in X$ for $\sigma \in \mathbb{R}^+$, evolving according to

$$z_\sigma(t + 1) = F_\sigma(t)z(t)$$

(6.4)

with $z_\sigma(0) = z(0)$ and $F_\sigma(t) \in \mathcal{B}(X)$ and converging pointwise in time for all $t$,

$$\lim_{\sigma \to \infty} \|z_\sigma(t) - z(t)\| = 0.$$  

(6.5)

Now, let $U_\sigma(t) \in \mathcal{B}(X)$ be a family of symmetric, positive definite operators, and let

$$V_\sigma(z_\sigma(t), t) = \langle z_\sigma(t), U_\sigma(t)z_\sigma(t) \rangle$$

(6.6)

be the associated quadratic forms.

Suppose there exist constants $T$ and $\eta$; bounded symmetric positive definite operators $W_\sigma(t), W(t), G(t)$; a set $S$; and real-valued function $M(\cdot)$ on $S$ such that

1. $V_\sigma(z_\sigma(t + 1), t + 1) - V_\sigma(z_\sigma(t), t) \leq 0$ and

$$V_\sigma(z_\sigma(t), t) - V_\sigma(z_\sigma(t - T), t - T) \leq \langle z_\sigma(t), W(t)z_\sigma(t) \rangle$$

2. $\liminf_{\sigma \to \infty} V_\sigma(z_\sigma(0), 0) < \infty \quad \forall z(0) \in S,$

3. $\langle z', U_\sigma(0)z' \rangle \leq M(z') \quad \forall z' \in S$

4. $\lim_{\sigma \to \infty} \langle z', W_\sigma(t)z' \rangle = \langle z', W(t)z' \rangle \quad \forall z' \in X$

5. $G^2(t) = W(t)$ and $\bigcap \bar{R}(G(t)) = X$, where the bar denotes closure in the strong topology

6. $\langle z', U_\sigma(t)z' \rangle \geq \eta \|z'\|^2$ for all $\sigma, t$ and all $z' \in X$.

Then,

$$\lim_{t \to \infty} \langle z(t), z' \rangle = 0 \quad \forall z(0) \in S, z' \in X. \quad (6.7)$$

Remarks. Of the six enumerated conditions of the theorem, the first ensures that the value of $V_\sigma$ descends along the trajectories of $z_\sigma$. The trajectories $z_\sigma$ approximate those of $z(t)$, which are of principal interest. The second and fourth conditions ensure that $V_\sigma$ and $W_\sigma$ are well-behaved as $\sigma$ varies. The third and sixth conditions guarantee that the quadratic forms associated with $U_\sigma$ are bounded above and below as $\sigma$ varies. These conditions, when combined with the first condition, can be used to bound the norm of $z(t)$. Lastly, the fifth condition places restrictions on the variability of $W_\sigma$ as a function of $t$. Specifically, the condition guarantees that the functional $\langle z, W(t)z \rangle$ provides information about the norm of $z(t)$. This condition is discussed further in the next section.

Proof. The proof is broken down into three steps.

1. The first step is to note that

$$0 \leq V_\sigma(z_\sigma(t), t) \leq V_\sigma(z_\sigma(0), 0) - \sum_{\tau = 0}^{t} \langle z_\sigma(\tau T + s), W_\sigma(\tau T + s)z_\sigma(\tau T + s) \rangle \leq V_\sigma(z_\sigma(0), 0) \quad \forall t. \quad (6.8)$$
2. The next step is to show that
\[
\lim_{t \to \infty} \langle z(t), W(t)z(t) \rangle = 0 \quad \forall z(0) \in S. \tag{6.9}
\]
To see this, note that by Fatou's lemma and (6.8),
\[
\sum_{\tau = 1}^{\infty} \inf_{\sigma \to \infty} \langle z_{\sigma}, W_{\sigma}(\tau T + s)z_{\sigma}(\tau T + s) \rangle \leq \liminf_{\sigma \to \infty} \sum_{\tau = 1}^{\infty} \langle z_{\sigma}, W_{\sigma}(\tau T + s)z_{\sigma}(\tau T + s) \rangle \leq \liminf_{\sigma \to \infty} V_{\sigma}(z_{\sigma}(0), 0). \tag{6.10}
\]
for any \( s \in [0, T) \). Thus,
\[
\sum_{\tau = 1}^{\infty} \langle z(\tau T + s), W(\tau T + s)z(\tau T + s) \rangle < \infty \tag{6.12}
\]
and (6.9) follows.

3. Lastly, weak stability is established. Now, one can rewrite (6.9) as
\[
\lim_{t \to \infty} ||G(t)z(t)||^2 = 0 \quad \forall z(0) \in S \tag{6.13}
\]
by Condition 5. That \( G(t)z(t) \) converges to zero strongly implies it converges to zero weakly, i.e.
\[
\lim_{t \to \infty} \langle G(t)z(t), z' \rangle = 0 \quad \forall z' \in X, z(0) \in S. \tag{6.14}
\]
Fix \( z' \in X \). Let \( \{z'_{m}\} \subseteq \bigcap R(G(t)) \) be a sequence converging to \( z' \) strongly. One can do this by Condition 5. Then,
\[
|\langle z(t), z' \rangle| = |\langle z(t), z' - z'_m + z'_m \rangle| \leq |\langle z(t), z'_m \rangle| + |\langle z(t), z' - z'_m \rangle| \tag{6.15}
\]
\[
\leq |\langle z(t), z'_m \rangle| + ||z(t)||\|z' - z'_m\| \tag{6.16}
\]
\[
\leq |\langle z(t), z'_m \rangle| + M(\langle z(0) \rangle) \eta \|z' - z'_m\| \tag{6.17}
\]
where the second to last step follows from the Cauchy-Schwartz inequality, and the last step follows from Conditions 3 and 6. Fix \( \varepsilon > 0 \). Then, there exists \( m_0 \) such that
\[
M(\langle z(0) \rangle) \eta \|z' - z'_m\| \leq \frac{\varepsilon}{2}. \tag{6.19}
\]
Moreover, there exists a \( t_0 \) such that
\[
|\langle z(t), z'_m \rangle| \leq \frac{\varepsilon}{2} \tag{6.20}
\]
for all \( t > t_0 \) and hence
\[
|\langle z(t), z' \rangle| \leq \varepsilon \tag{6.21}
\]
for all \( t > t_0 \). Since this holds for all \( \varepsilon > 0 \),
\[
\lim_{t \to \infty} |\langle z(t), z' \rangle| = 0 \quad \forall z' \in S. \tag{6.22}
\]
7. Weak Stability of the Kalman Filter for Space-time Estimation. Theorem 6.3 can be used to establish weak stability of the Kalman filter when the observability condition (5.3) of Theorem 5.1 is weakened. The condition (5.3) is replaced with two mild conditions on the observability Gramian $\mathcal{I}(t, s)$. The first requirement is that there exists a $T$ for which $\mathcal{I}(t, t - T)$ is positive definite (but not necessarily uniformly positive definite) for all $t \geq T$. The other principal condition is that

$$\bigcap_{t \geq T} \mathcal{R}\left(\mathcal{I}(t, t - T)\right)^{1/2} = X$$  \hfill (7.1)

hold. This is a fairly mild condition. In particular, the range of $(\mathcal{I}(t, t - T))^{1/2}$ is dense if $\mathcal{I}(t, t - T)$ is positive definite [30, Theorem 12.12b]. Moreover, the ranges of $(\mathcal{I}(t, t - T))^{1/2}$, for all times $t$, will overlap significantly in many cases.

Specifically, the ranges will overlap significantly if the state-space model is quasi-periodic so that there exists a $T$ for which $(\mathcal{I}(t, t - T))^{1/2}$ does not change significantly as $t$ varies. This is the case in many satellite remote sensing scenarios, which, as mentioned in the introduction, have motivated the stability investigation in this paper.

**Theorem 7.1.** Suppose that there exist constants $\beta_2, \gamma_1, \gamma_2 > 0$ such that

$$0 < \mathcal{I}(t, t - T) \leq \beta_2 I \quad \forall t > T$$ \hfill (7.2)

$$\gamma_1 I \leq A(t) \leq \gamma_2 I \quad \forall t$$ \hfill (7.3)

and that the square-root of $\mathcal{I}(t, t - T)$ satisfies

$$\bigcap_{t \geq T} \mathcal{R}\left(\mathcal{I}(t, t - T)\right)^{1/2} = X.$$  \hfill (7.4)

Then, the dynamics of the Kalman filter are weakly stable, i.e.

$$\lim_{t \to \infty} \langle z(t), z' \rangle = 0 \quad \forall z' \in X$$  \hfill (7.5)

where $z(t)$ obeys the dynamics (2.10).

**Proof.** The proof of Theorem 7.1 makes use of the same sequence of systems used in the proof of Theorem 5.1.

The differences between the proofs of Theorems 5.1 and 7.1 start with the bounds on the update error covariance for the modified system. Specifically, by Theorem 3.2, the update error covariance of the modified system, $\Lambda'_{r,\sigma}(t|t)$, is bounded below as follows:

$$\left(\frac{\gamma_1}{\sigma_t\gamma_1 + (T + 1)\sigma_t^3(\alpha_2^2\beta_2^2\gamma_2^2 + \beta_2)}\right)^2 I \leq \Lambda'_{r,\sigma}(t|t).$$  \hfill (7.6)

Moreover, the error covariance $\Lambda'_{r,\sigma}(t|t)$ is always bounded above by the prior covariance of the state. Hence, one can consider using

$$V_r(x, t) = \langle x, (\Lambda'_{r,\sigma}(t|t))^{-1} x \rangle \quad t \geq T$$  \hfill (7.7)

to establish weak stability of the Kalman filter.

In order to use this sequence of systems to establish weak stability of the original filter dynamics (2.10), one needs to verify each of the six enumerated conditions of Theorem 6.3 for $U_r(t) = (\Lambda'_{r,\sigma}(t|t))^{-1}$, $U = \Lambda^{-1}(T|T)$, $W_r(t) = W = \mathcal{I}(t, t - T)$, $G(t) = (\mathcal{I}(t, t - T))^{1/2}$, $M(z') = \max_{\sigma \geq 1}(z', (\Lambda'_{r,\sigma}(T|T))^{-1}, z')$, and $\eta = \beta_2$. The verification follows, where, as in the proof of Theorem 5.1, the trajectories are examined starting at time $T$, not 0.
1. As in the proof of Theorem 5.1,
\begin{align}
V_\sigma(z_\sigma(t+1),t+1) - V_\sigma(z_\sigma(t),t) & \leq 0 \quad (7.8) \\
V_\sigma(z_\sigma(t),t) - V_\sigma(z_\sigma(t-T),t-T) & \leq \langle z(t), I(t,t-T)z(t) \rangle. \quad (7.9)
\end{align}

2. Also for the same reasons as in the proof of Theorem 5.1,
\[ \lim_{\sigma \to \infty} \langle z(T), (\Lambda^\iota_{\sigma}(T|T))^{-1}z(T) \rangle = \langle z(T), \Lambda^{-1}(T|T)z(T) \rangle \quad (7.10) \]
and \( z(T) \in D(\Lambda^{-1}(T|T)) \) for all initial conditions \( z(0) \).

3. Now,
\[ \max_{\sigma \geq 1} \langle z', (\Lambda'_{\sigma}(T|T))^{-1}, z' \rangle \]
exists for all \( z' \in D(\Lambda^{-1}(T|T)) \) by continuity of \( (\Lambda'_{\sigma})^{-1} \) as a function of \( \sigma \) and the
existence of a limit as \( \sigma \) tends to infinity. Thus,
\[ \langle z', (\Lambda'_{\sigma}(T|T))^{-1}z' \rangle \leq \max_{\sigma \geq 1} \langle z', (\Lambda'_{\sigma}(T|T))^{-1}, z' \rangle \quad \forall z' \in D(\Lambda^{-1}(T|T)). \quad (7.12) \]

4. \( \lim_{\sigma \to \infty} \langle z', I(t,t-T)z' \rangle = \langle z', I(t,t-T)z' \rangle \) identically.
5. \( \int_{\beta \geq T} R((I(t,t-T))^1/2) = X \) by the assumptions of Theorem 7.1.
6. \( \langle z', (\Lambda'_{\sigma}(t|t))^{-1}z' \rangle \geq \beta_\epsilon \|z'\|^2 \) for all \( t \geq T \) by (7.6).

Now, the desired result follows from Theorem 6.3. \( \square \)

8. Conclusions. The set of theorems presented in this paper establish stability of the Kalman filter under conditions mild enough so that they apply to scenarios arising in remote sensing. In particular, the state-space models in [31,32] have positive definite driving noise and measurement structures such that the observability Gramian is uniformly positive definite. The results in this paper then allow us to conclude that the Kalman filters for these problems are strongly stable. If the measurements were of poorer quality but still leading to a positive definite observability Gramian, the results of this paper guarantee that the corresponding Kalman filter would be weakly stable.

Establishing conditions under which the Kalman filter exhibits the different forms of stability has necessitated the development of an appropriate Lyapunov theory. In particular, Theorems 4.5 and 6.3 provide sufficient conditions for a linear system to be strongly and weakly stable, respectively. Moreover, the conditions in Theorem 4.5 are sufficient to guarantee that the system is strongly summable, and Theorem 4.6 establishes that these conditions are also necessary.

One topic left for future research is an examination of the robustness of the Kalman filter. Specifically, if the dynamics (2.10) are forced by a bounded perturbation at every time, one is interested in how the filtered estimates behave. We conjecture that they grow, but slowly. Another area of research is establishing conditions for well-defined steady-state behavior of the Kalman filter error covariance when the underlying state-space model is time-invariant. Although Hager and Horowitz [9] address the issue of steady-state error covariance behavior, they restrict their discussion to scenarios less general than those considered here.

Appendix. Proof of Theorem 3.2. A complete proof of Theorem 3.2 follows. The outline of the proof is similar to that of [8] but takes into account errors cited
in [7] and incorporates the assumptions in Theorem 3.2 concerning the boundedness of $A(t)$ and its inverse.

**Proof.** Consider the system

\begin{align}
\dot{x}'(t + 1) &= A^{-*}x'(t) + A^{-*}(t)C^*(t)w'(t) \\
y'(t) &= \dot{x}'(t) + n'(t)
\end{align}

(1.1)

(1.2)

where $\text{Cov}(w'(t)) = \Lambda_\omega^{-1}(t)$, $\text{Cov}(n'(t)) = \Lambda_{\omega'}^{-1}(t - 1)$, and the system is initialized at time 0 with no measurement and $\Lambda_{\omega'} = \Lambda_{\omega}^{-1}$. The reachability and observability Gramians for this system are

\begin{align}
\mathcal{R}'(t, s) &= \sum_{\tau=s}^{t-1} \Phi^{-1}(t, \tau + 1)\Phi^{-1}(\tau + 1, \tau)C^*(\tau)\Lambda_\omega^{-1}(\tau)C(\tau)\Phi^{-1}(\tau + 1, \tau)\Phi^{-1}(t, \tau + 1) \\
&= A^{-*}(t-1)I(t-1, s)A^{-*}(t-1), \\
\mathcal{I}'(t, s) &= \sum_{\tau=s}^{t} \Phi^{-1}(\tau, \tau)\Lambda_{\omega'}^{-1}(\tau - 1)\Phi^{-1}(\tau, t).
\end{align}

(3.1)

(3.2)

One can rewrite the reachability Gramian as

\begin{align}
\mathcal{R}'(t, s) &= A^{-*}(t-1)\left(\sum_{\tau=s}^{t-1} \Phi^{*}(\tau, t - 1)C^{*}(\tau)\Lambda_\omega^{-1}(\tau)C(\tau)\Phi(t, t - 1)\right)A^{-1}(t-1) \\
&= A^{-*}(t-1)I(t-1, s)A^{-*}(t-1), \\
&= \mathcal{R}(t, s - 1).
\end{align}

(5.1)

and the observability Gramian as

\begin{align}
\mathcal{I}'(t, s) &= \sum_{\tau=s}^{t} \Phi(\tau, \tau)\Lambda_{\omega'}^{-1}(\tau - 1)\Phi^{*}(\tau, t) \\
&= \sum_{\tau=s+1}^{t-1} \Phi(\tau, \tau + 1)\Lambda_{\omega'}^{-1}(\tau)\Phi^{*}(t, \tau + 1) \\
&= \mathcal{R}(t, s - 1).
\end{align}

(6.1)

By (3.4), (3.5), and (3.6),

\begin{align}
\frac{\beta_1}{\gamma_2}I &\leq \mathcal{R}'(t, t - T) \leq \frac{\beta_2}{\gamma_1}I & \forall t \geq T \\
\alpha_1 &\leq \mathcal{I}'(t, t - T) \leq \alpha_2 I & \forall t \geq T.
\end{align}

(7.1)

(8.1)

Theorem 3.2 then implies that the error covariance of the new system satisfies

\begin{align}
\Lambda'_e(t|t) &\leq \left(\frac{1}{\alpha_1} + T\frac{\alpha_2^2\beta_2}{\alpha_1^2\gamma_1}\right)I,
\end{align}

(9.1)

and, thus,

\begin{align}
(\Lambda'_e)^{-1}(t|t) &\geq \left(\frac{\alpha_1^2\gamma_1}{\alpha_1^2\gamma_1 + T\alpha_2^2\beta_2}\right)I.
\end{align}

(10.1)
Now, the recursions for the Kalman filter error covariances for the new system are

\[
(A'_e)^{-1}(t|t) = (A'_e)^{-1}(t|t-1) + A_w(t-1) \\
A'_e(t+1|t) = A^{-1}A'_e(t|t)A^{-1}(t) + A^{-1}C(t)A^{-1}(t)
\] (A.11) (A.12)

Comparing this with the recursions for \( A_e(t|t) \), the Kalman filter error covariances of the original system, one notices that

\[
A_e^{-1}(t|t) = (A'_e)^{-1}(t|t) + C^*(t)A_e^{-1}(t)C(t) + C^*(t)A_e^{-1}(t)C(t).
\] (A.13)

By (A.10) and (3.6),

\[
A_e(t|t) = (A'_e)^{-1}(t|t) \geq \left( \frac{\alpha_1^2 \gamma_1}{\alpha_1 \gamma_1 + T \alpha_2^2 \beta_2} + \beta_2 \right) I.
\] (A.14)

**REFERENCES**


